L Number	Hits	Search Text	DB	Time stamp
1	36	(linear adj operator) with equation	USPAT	2004/07/06 12:46
2	546022	physical	USPAT	2004/07/06 12:09
3	8	((linear adj operator) with equation) and physical	USPAT	2004/07/06 11:55
4	1580688	structure	USPAT	2004/07/06 12:09
5	22	((linear adj operator) with equation) and structure	USPAT	2004/07/06 12:09
6	337	linear adj operator	USPAT	2004/07/06 12:54
7	22	((linear adj operator) with equation) and structure	USPAT	2004/07/06 12:46
8	8	((linear adj operator) with equation) and physical	USPAT	2004/07/06 12:46
18	72511	physical with (structure or composition)	USPAT	2004/07/06 13:00
19	14	(linear adj operator) and (physical with (structure or composition))	USPAT	2004/07/06 12:54
20	1901851	structure or composition	USPAT	2004/07/06 13:00
21	211	(linear adj operator) and (structure or composition)	USPAT	2004/07/06 13:00
22	2161840	cable or line or conductor	USPAT	2004/07/06 13:01
23	173	((linear adj operator) and (structure or composition)) and (cable or line or conductor)	USPAT	2004/07/06 13:01
-	7353	(module or add-on or (sub with circuit$5)) with (id or identification)	USPAT	2004/07/06 11:55
-	2083334	sub-routine or subroutine or application	USPAT	2004/07/06 07:37
-	1046	((module or add-on or (sub with circuit$5)) with (id or identification)) same (sub-routine or subroutine or application)	USPAT	2004/07/06 07:38
-	9452	mainroutine or (main adj (program or application))	USPAT	2004/07/06 07:39
-	7	(((module or add-on or (sub with circuit$5)) with (id or identification)) same (sub-routine or subroutine or application)) same (mainroutine or (main adj (program or application)))	USPAT	2004/07/06 07:40
-	74	(((module or add-on or (sub with circuit$5)) with (id or identification)) same (sub-routine or subroutine or application)) and (mainroutine or (main adj (program or application)))	USPAT	2004/07/06 07:41
-	3	stimson.in. and 379/$.ccls.	USPAT	2004/07/06 09:57
-	337	linear adj operator	USPAT	2004/07/06 11:26
-	2111873	cable or line or trunk	USPAT	2004/07/06 11:26
-	40477	physical with structure	USPAT	2004/07/06 11:26
-	2880	(cable or line or trunk) same (physical with structure)	USPAT	2004/07/06 11:04
-	0	(linear adj operator) same ((cable or line or trunk) same (physical with structure))	USPAT	2004/07/06 11:04
-	1	(linear adj operator) and ((cable or line or trunk) same (physical with structure))	USPAT	2004/07/06 11:09
-	11	(linear adj operator) and (cable or line or trunk) and (physical with structure)	USPAT	2004/07/06 11:14
-	1927062	cable or line or trunk	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:23
-	61	linear adj operator	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:23
-	5064	physical with structure	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
-	0	(cable or line or trunk) and (linear adj operator) and (physical with structure)	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
-	0	(linear adj operator) and (physical with structure)	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
-	8	(cable or line or trunk) and (linear adj operator)	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
-	121	linear adj operator	US-PGPUB	2004/07/06 11:26
-	393216	cable or line or trunk	US-PGPUB	2004/07/06 11:26
-	13505	physical with structure	US-PGPUB	2004/07/06 11:26
-	5	(linear adj operator) and (cable or line or trunk) and (physical with structure)	US-PGPUB	2004/07/06 11:26

L Number	Hits	Search Text	DB	Time stamp
1	337	linear adj operator	USPAT	2004/07/06 11:26
2	2111873	cable or line or trunk	USPAT	2004/07/06 11:26
3	40477	physical with structure	USPAT	2004/07/06 11:26
4	2880	(cable or line or trunk) same (physical with structure)	USPAT	2004/07/06 11:04
5	0	(linear adj operator) same ((cable or line or trunk) same (physical with structure))	USPAT	2004/07/06 11:04
6	1	(linear adj operator) and ((cable or line or trunk) same (physical with structure))	USPAT	2004/07/06 11:09
7	11	(linear adj operator) and (cable or line or trunk) and (physical with structure)	USPAT	2004/07/06 11:14
8	1927062	cable or line or trunk	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:23
9	61	linear adj operator	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:23
10	5064	physical with structure	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
11	0	(cable or line or trunk) and (linear adj operator) and (physical with structure)	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
12	0	(linear adj operator) and (physical with structure)	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
13	8	(cable or line or trunk) and (linear adj operator)	EPO; JPO; DERWENT; IBM_TDB	2004/07/06 11:24
14	121	linear adj operator	US-PGPUB	2004/07/06 11:26
15	393216	cable or line or trunk	US-PGPUB	2004/07/06 11:26
16	13505	physical with structure	US-PGPUB	2004/07/06 11:26
17	5	(linear adj operator) and (cable or line or trunk) and (physical with structure)	US-PGPUB	2004/07/06 11:26

=> d hit all 1-5

L4 ANSWER 1 OF 5 INSPEC (C) 2004 IEE on STN

AB Exploring the possible **links** between the mathematical field of fractional calculus and the electromagnetic theory has been one of the topics of our research interests. We have studied the possibility of bringing the tools of fractional calculus and electromagnetic theory together, and have explored and developed the topic of fractional paradigm in electromagnetic theory. Fractional calculus is a branch of mathematics that addresses the mathematical properties of operation of fractional differentiation and fractional integration operators involving derivatives and integrals to arbitrary non-integer orders. We have applied the tools of fractional calculus in various problems in electromagnetic fields and waves, and have obtained interesting results that highlight certain notable features and promising potential applications of these operators in electromagnetic theory. Moreover, since fractional derivatives/integrals are effectively the result of fractionalization of differentiation and integration operators, we have investigated the notion of fractionalization of some other **linear operators** in electromagnetic theory. Searching for such operator fractionalization has led us to interesting solutions in radiation and scattering problems.

CT APPROXIMATION THEORY; DIFFERENTIATION; ELECTROMAGNETIC WAVE SCATTERING; FRACTALS; INTEGRATION; MATHEMATICAL OPERATORS; **PHYSICAL** OPTICS

ST fractionalization methods; EM radiation problems; EM scattering problems; fractional calculus; electromagnetic theory; fractional paradigm; fractional differentiation operators; fractional integration operators; electromagnetic fields; electromagnetic waves; **linear operators**; **physical optics approximation**

AN 2001:6905565 INSPEC DN A2001-10-4110H-023; B2001-05-5210-037

TI Fractionalization methods and their applications to radiation and scattering problems.

AU Engheta, N. (Moore Sch. of Electr. Eng., Pennsylvania Univ., Philadelphia, PA, USA)

SO Conference Proceedings 2000 International Conference on Mathematical Methods in Electromagnetic Theory (Cat. No.00EX413)
Piscataway, NJ, USA: IEEE, 2000. p.34-40 vol.1 of 2 vol. 719 pp. 13 refs.
Conference: Kharkov, Ukraine, 12-15 Sept 2000
Sponsor(s): IEEE AP/MTT/AES/ED/GRS/LEO Societies East Ukraine Joint Chapter; Ukrainian Nat. URSI Committee; Sci. Council of NAS on Radio Phys. & Microwave Electron.; Inst. Radio-Phys. & Electron. NAS; Inst. Radio Aston. NAS; Kharkov Nat. Univ.; IEEE AP, MTT & ED Soc.; URSI
ISBN: 0-7803-6347-7

DT Conference Article

TC Theoretical

CY United States

LA English

AB Exploring the possible **links** between the mathematical field of fractional calculus and the electromagnetic theory has been one of the topics of our research interests. We have studied the possibility of bringing the tools of fractional calculus and electromagnetic theory together, and have explored and developed the topic of fractional paradigm in electromagnetic theory. Fractional calculus is a branch of mathematics that addresses the mathematical properties of operation of fractional differentiation and fractional integration operators involving derivatives and integrals to arbitrary non-integer orders. We have applied the tools of fractional calculus in various problems in electromagnetic fields and waves, and have obtained interesting results that highlight certain notable features and promising potential applications of these operators in electromagnetic theory. Moreover, since fractional derivatives/integrals are effectively the result of fractionalization of differentiation and integration operators, we have investigated the notion of fractionalization of some other **linear operators** in electromagnetic theory. Searching for such operator fractionalization has led us to interesting solutions in radiation and scattering problems.

CC A4110H Electromagnetic waves: theory; A0260 Numerical approximation and analysis; B5210 Electromagnetic wave propagation; B0290M Numerical integration and differentiation; B0290F Interpolation and function approximation (numerical analysis)
CT APPROXIMATION THEORY; DIFFERENTIATION; ELECTROMAGNETIC WAVE SCATTERING; FRACTALS; INTEGRATION; MATHEMATICAL OPERATORS; **PHYSICAL** OPTICS
ST fractionalization methods; EM radiation problems; EM scattering problems; fractional calculus; electromagnetic theory; fractional paradigm; fractional differentiation operators; fractional integration operators; electromagnetic fields; electromagnetic waves; **linear operators**; **physical optics approximation**

L4 ANSWER 2 OF 5 INSPEC (C) 2004 IEE on STN
AB The family of Gowdy universes with the spatial topology of a three-torus is studied both classically and quantum mechanically. Starting with the Ashtekar formulation of Lorentzian general relativity, we introduce a gauge fixing procedure to remove almost all of the nonphysical degrees of freedom. In this way, we arrive at a reduced model that is subject only to one homogeneous constraint. The phase space of this model is described by means of a canonical set of elementary variables. These are two real, homogeneous variables and the Fourier coefficients for four real fields that are periodic in the angular coordinate which does not correspond to a Killing field of the Gowdy spacetimes. We also obtain the explicit expressions for the **line** element and reduced Hamiltonian. We then proceed to quantize the system by representing the elementary variables as **linear operators** acting on a vector space of analytic functionals. The inner product on that space is selected by imposing Lorentzian reality conditions. We find the quantum states annihilated by the operator that represents the homogeneous constraint of the model and construct with them the Hilbert space of **physical** states. Finally, we derive the general form of the quantum observables of the model.
ST Gowdy universe; three-torus; spatial topology; Lorentzian general relativity; gauge fixing; phase space; Fourier coefficients; angular coordinate; Gowdy spacetimes; **line element**; reduced Hamiltonian; canonical quantization; quantum states; Hilbert space; **physical states**; quantum observables
AN 1997:5656020 INSPEC DN A9718-9880-025
TI Canonical quantization of the Gowdy model.
AU Marugan, G.A.M. (Inst. de Matematicas y Fisica Fundamental, CSIC, Madrid, Spain)
SO Physical Review D (15 July 1997) vol.56, no.2, p.908-19. 23 refs.
Doc. No.: S0556-2821(97)04514-1
Published by: APS through AIP
Price: CCCC 0556-2821/97/56(2)/908(12)/$10.00
CODEN: PRVDAQ ISSN: 0556-2821
SICI: 0556-2821(19970715)56:2L.908:CQGM;1-#
DT Journal
TC Theoretical
CY United States
LA English
AB The family of Gowdy universes with the spatial topology of a three-torus is studied both classically and quantum mechanically. Starting with the Ashtekar formulation of Lorentzian general relativity, we introduce a gauge fixing procedure to remove almost all of the nonphysical degrees of freedom. In this way, we arrive at a reduced model that is subject only to one homogeneous constraint. The phase space of this model is described by means of a canonical set of elementary variables. These are two real, homogeneous variables and the Fourier coefficients for four real fields that are periodic in the angular coordinate which does not correspond to a Killing field of the Gowdy spacetimes. We also obtain the explicit expressions for the **line** element and reduced Hamiltonian. We then proceed to quantize the system by representing the elementary variables as **linear operators** acting on a vector space

of analytic functionals. The inner product on that space is selected by imposing Lorentzian reality conditions. We find the quantum states annihilated by the operator that represents the homogeneous constraint of the model and construct with them the Hilbert space of **physical** states. Finally, we derive the general form of the quantum observables of the model.

CC A9880D Theoretical cosmology; A0240 Geometry, differential geometry, and topology; A0420C Fundamental problems and general formalism in general relativity; A0460 Quantum theory of gravitation

CT COSMOLOGY; GENERAL RELATIVITY; QUANTISATION (QUANTUM THEORY); QUANTUM GRAVITY; SPACE-TIME CONFIGURATIONS

ST Gowdy universe; three-torus; spatial topology; Lorentzian general relativity; gauge fixing; phase space; Fourier coefficients; angular coordinate; Gowdy spacetimes; **line element**; reduced Hamiltonian; canonical quantization; quantum states; Hilbert space; **physical states**; quantum observables

L4 ANSWER 3 OF 5 INSPEC (C) 2004 IEE on STN

AB This paper uses a wavelet transform approach to actively image an object continuously distributed in range and velocity. It is shown that by transmitting a high resolution, i.e. large time-bandwidth product, signal into the environment and operating on the echo with a wavelet transform, an estimate of the delay-time-scale representation or wideband spreading function of the object can be obtained. The wideband spreading function (WBSF) is a characterization of the time-varying propagation/scattering associated with the channel being imaged. It is shown that the **linear operator** that acts on the wideband spreading function to form the echo is in the form of an inverse wavelet transform and the adjoint operator is in the form of a forward wavelet transform. Thus, the wavelet transform is a natural transform for the investigation of wideband spreading functions. By combining information extracted from wavelet estimates of the WBSF associated with independently located sensors, it is possible to estimate vectors which describe the **physical** characteristics of the object in the channel. Specifically, the support curve of the WBSF in the wavelet domain can be directly related to the projections of these vectors along the **line** of sight of each of the sensors. Therefore, it is necessary to obtain a wavelet estimate of the WBSF which highly resolves the support curve. The wavelet transform estimate is shown to be limited by the resolution capabilities of the auto-wavelet transform of the transmitted signal therefore establishing the need for high resolution transmit signals.

ST wavelet processing; continuously distributed objects; object imaging; large time-bandwidth product signal; echo; delay-time-scale representation; wideband spreading function; time-varying propagation; time-varying scattering; inverse wavelet transform; adjoint operator; forward wavelet transform; **sensor line of sight**; auto-wavelet transform; high resolution transmit signals; rough rotating sphere; wideband/narrowband signals; remote sensing

AN 1996:5201507 INSPEC DN B9604-6140C-235

TI Wavelet processing applied to the estimation of continuously distributed objects.

AU Dixon, T.L. (Appl. Res. Lab., Pennsylvania State Univ., State College, PA, USA)

SO Proceedings of the SPIE - The International Society for Optical Engineering (1995) vol.2569, pt.1, p.164-74. 19 refs.
Published by: SPIE-Int. Soc. Opt. Eng
Price: CCCC 0 8194 1928 1/95/$6.00
CODEN: PSISDG ISSN: 0277-786X
SICI: 0277-786X(1995)2569:1L.164:WPAE;1-5
Conference: Wavelet Applications in Signal and Image Processing III. San Diego, CA, USA, 12-14 July 1995
Sponsor(s): SPIE

DT Conference Article; Journal

TC Theoretical
CY United States
LA English
AB This paper uses a wavelet transform approach to actively image an object continuously distributed in range and velocity. It is shown that by transmitting a high resolution, i.e. large time-bandwidth product, signal into the environment and operating on the echo with a wavelet transform, an estimate of the delay-time-scale representation or wideband spreading function of the object can be obtained. The wideband spreading function (WBSF) is a characterization of the time-varying propagation/scattering associated with the channel being imaged. It is shown that the **linear operator** that acts on the wideband spreading function to form the echo is in the form of an inverse wavelet transform and the adjoint operator is in the form of a forward wavelet transform. Thus, the wavelet transform is a natural transform for the investigation of wideband spreading functions. By combining information extracted from wavelet estimates of the WBSF associated with independently located sensors, it is possible to estimate vectors which describe the **physical** characteristics of the object in the channel. Specifically, the support curve of the WBSF in the wavelet domain can be directly related to the projections of these vectors along the **line** of sight of each of the sensors. Therefore, it is necessary to obtain a wavelet estimate of the WBSF which highly resolves the support curve. The wavelet transform estimate is shown to be limited by the resolution capabilities of the auto-wavelet transform of the transmitted signal therefore establishing the need for high resolution transmit signals.
CC B6140C Optical information, image and video signal processing; B0290Z Other numerical methods; B6310 Radar theory
CT ECHO; ELECTROMAGNETIC WAVE SCATTERING; IMAGE REPRESENTATION; INVERSE PROBLEMS; PARAMETER ESTIMATION; RADAR IMAGING; REMOTE SENSING; SIGNAL RESOLUTION; TIME-VARYING CHANNELS; WAVELET TRANSFORMS
ST wavelet processing; continuously distributed objects; object imaging; large time-bandwidth product signal; echo; delay-time-scale representation; wideband spreading function; time-varying propagation; time-varying scattering; inverse wavelet transform; adjoint operator; forward wavelet transform; **sensor line of sight**; auto-wavelet transform; high resolution transmit signals; rough rotating sphere; wideband/narrowband signals; remote sensing

L4 ANSWER 4 OF 5 INSPEC (C) 2004 IEE on STN
AB The authors describe a multivariable deconvolution method which approximates the unknown kernel by a finite series of Laguerre functions (which define a complete basis in L/sup 2/(0, infinity)). The possibility of describing the input-output relation by a convolution with a L/sup 2/(0, infinity) kernel is justified by some sufficient conditions (based upon the properties of **linear operators**). The algorithm which is then presented uses least squares as a criterion and indications are given for a possible on-**line** procedure. At last, a numerical example operating on real **physical** data identification of a part of a steam generator illustrates the algorithm.
AN 1972:374076 INSPEC DN C72008611
TI A deterministic identification method of linear stationary dynamic systems.
AU Borget, G.; Faure, P.
SO Bulletin de la Direction des Etudes et Recherches, Serie C (1971) no.2, p.5-31. 17 refs.
CODEN: EDBCAA ISSN: 0013-4511
DT Journal
TC Theoretical
CY France
LA French
AB The authors describe a multivariable deconvolution method which approximates the unknown kernel by a finite series of Laguerre functions

(which define a complete basis in L/sup 2/(0, infinity)). The possibility of describing the input-output relation by a convolution with a L/sup 2/(0, infinity) kernel is justified by some sufficient conditions (based upon the properties of **linear operators**). The algorithm which is then presented uses least squares as a criterion and indications are given for a possible on-**line** procedure. At last, a numerical example operating on real **physical** data identification of a part of a steam generator illustrates the algorithm.

CC C1220 Simulation, modelling and identification
CT HEAT SYSTEMS; IDENTIFICATION; LINEAR SYSTEMS
ST linear stationary dynamic systems; multivariable deconvolution method; Laguerre functions; identification; steam generator

L4 ANSWER 5 OF 5 INSPEC (C) 2004 IEE on STN

AB Conventional filters are linear time-invariant systems and operate with respect to the cisoidal orthonormal basis, parameterized by frequency, the cisoidal functions being the characters of the real group (isomorphic to real time). Semiconductor technology makes competitive the use of linear dyadic-invariant filters, which operate with respect to the Walsh-function orthonormal basis, parameterized by sequence, the Walsh functions being characters of the dyadic group. The binary digital computer is intimately related to a finite sub-group of the dyadic group, so digital filtering is well implemented by means of Walsh functions. The related **linear operator** called the logical derivative, is briefly discussed. References are made to work on these **lines**, both theoretical and practical, in America and in continental Europe, and to other applications of Walsh functions in communication engineering. The report ends with a speculation on the dyadic group as a model of the temporal domain that might allow a reformulation of **physical** laws to match them better to the abilities of the digital computer.

AN 1970:196972 INSPEC DN C70021107
TI Digital filtering in dyadic-time and sequency.
AU Gibbs, J.E.
CS Nat. Phys. Lab., Teddington, UK
NR DES Rept. No.5
SO 1970. 16 pp. 63 refs.
DT Report
CY United Kingdom
LA English

AB Conventional filters are linear time-invariant systems and operate with respect to the cisoidal orthonormal basis, parameterized by frequency, the cisoidal functions being the characters of the real group (isomorphic to real time). Semiconductor technology makes competitive the use of linear dyadic-invariant filters, which operate with respect to the Walsh-function orthonormal basis, parameterized by sequence, the Walsh functions being characters of the dyadic group. The binary digital computer is intimately related to a finite sub-group of the dyadic group, so digital filtering is well implemented by means of Walsh functions. The related **linear operator** called the logical derivative, is briefly discussed. References are made to work on these **lines**, both theoretical and practical, in America and in continental Europe, and to other applications of Walsh functions in communication engineering. The report ends with a speculation on the dyadic group as a model of the temporal domain that might allow a reformulation of **physical** laws to match them better to the abilities of the digital computer.

CC C1260 Information theory
CT FILTERING AND PREDICTION THEORY

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(FILE 'HOME' ENTERED AT 11:50:02 ON 06 JUL 2004)

FILE 'STNGUIDE' ENTERED AT 11:50:27 ON 06 JUL 2004

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     FILE 'HOME' ENTERED AT 11:50:32 ON 06 JUL 2004

     FILE 'INSPEC' ENTERED AT 11:50:51 ON 06 JUL 2004
L1            0 S (LINEAR(W)OPERATOR) AND (PHYSICAL(A)STRUCTURE)
L2           84 S (LINEAR(W)OPERATOR) AND (PHYSICAL)
L3       689298 S CABLE OR LINE OR LINK OR TRUNK
L4            5 S L2 AND L3
```

IEEE Xplore™ ◂ Back to Previous Page

Results Key:
JNL = Journal or Magazine **CNF** = Conference **STD** = Standard

1 **Transfer function models of multidimensional physical systems**
Rabenstein, R.;
Multidimensional Systems: Problems and Solutions (Ref. No. 1998/225), IEE Colloquium on , 14 Jan. 1998
Pages:1/1 - 1/7

IEE CNF

2 **Stationary variational expressions for radiated and scattered acoustic power and related quantities**
Pierce, A.;
Oceanic Engineering, IEEE Journal of , Volume: 12 , Issue: 2 , April 1987
Pages:404 - 411

IEEE JNL

3 **Physical preconditioning for modeling 2D large periodic arrays**
Capolino, F.; Wilton, D.R.; Jackson, D.R.;
Antennas and Propagation Society International Symposium, 2002. IEEE , Volume: 1 , 16-21 June 2002
Pages:512 - 515 vol.1

IEEE CNF

4 **Quadratic detectors for energy estimation**
Jing Fang; Atlas, L.E.;
Signal Processing, IEEE Transactions on [see also Acoustics, Speech, and Signal Processing, IEEE Transactions on] , Volume: 43 , Issue: 11 , Nov. 1995
Pages:2582 - 2594

IEEE JNL

5 **Nonstationary smoothing and prediction using network theory concepts**
Darlington, S.;
Information Theory, IEEE Transactions on , Volume: 5 , Issue: 5 , May 1959
Pages:1 - 13

IEEE JNL

6 **Concurrent complementary operator boundary conditions for optical beam propagation**
Law, C.T.; Zhang, X.;
Photonics Technology Letters, IEEE , Volume: 12 , Issue: 1 , Jan. 2000
Pages:56 - 58

IEEE JNL

7 **Transmission loss compensation in multiple transaction networks**
Shu Tao; Gross, G.;
Power Systems, IEEE Transactions on , Volume: 15 , Issue: 3 , Aug. 2000
Pages:909 - 915

IEEE JNL

8 **Fractionalization methods and their applications to radiation and scattering problems**
Engheta, N.;
Mathematical Methods in Electromagnetic Theory, 2000. MMET 2000. International Conference on , Volume: 1 , 12-15 Sept. 2000
Pages:34 - 40 vol.1

IEEE CNF

9 **SPICES1: A new Vlasov-Fokker-Planck simulation code**
Riccardo, V.; Coppa, G.G.M.; Lapenta, G.;
Plasma Science, 1997. IEEE Conference Record - Abstracts., 1997 IEEE International Conference on , 19-22 May 1997
Pages:323

IEEE CNF

10 **Analysis of delay circuits and systems using operators and causality**
Davis, A.M.;
Circuits and Systems, 1996. ISCAS '96., 'Connecting the World'., 1996 IEEE International Symposium on , Volume: 3 , 12-15 May 1996
Pages:13 - 16 vol.3

IEEE CNF

11 **An effective solution of electromagnetic integral equations in the boundary value problems of millimeter wave diffraction based on atomic functions**
Kravchenko, V.F.; Zamyatin, A.A.;
Physics and Engineering of Millimeter and Submillimeter Waves, 1998. MSMW '98. Third International Kharkov Symposium , Volume: 1 , 15-17 Sept. 1998
Pages:413 - 415 vol.1

IEEE CNF

12 **Automatic emittance measurement at the ATF**
Wang, X.J.; Malone, R.; Batchelor, K.; Ben-Zvi, I.;
Particle Accelerator Conference, 1993., Proceedings of the 1993 , 17-20 May 1993
Pages:2486 - 2488 vol.3

IEEE CNF

13 **Thermal stabilization of low level RF distribution systems at SLAC**
McCormick, D.; Ross, M.; Himel, T.; Spencer, N.;
Particle Accelerator Conference, 1993., Proceedings of the 1993 , 17-20 May 1993
Pages:1975 - 1977 vol.3

IEEE CNF

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14 **Adaptive linearization schemes for weakly nonlinear systems using adaptive linear and nonlinear FIR filters**
Gao, X.Y.; Snelgrove, W.M.;
Circuits and Systems, 1990., Proceedings of the 33rd Midwest Symposium on , 12-14 Aug. 1990
Pages:9 - 12 vol.1

IEEE CNF

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15 **A digital control system for accelerator operator-host computer interface**
Tilbrook, I.;
Particle Accelerator Conference, 1989. 'Accelerator Science and Technology'., Proceedings of the 1989 IEEE , 20-23 March 1989
Pages:1672 - 1674 vol.3

IEEE CNF

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[x] I [x] [x] previc [x] conter
Next: Linear Operators - II **Up:** Linear Operators - Basic **Previous:** Inverse of an Operator **Contents**

Eigen-values and Eigen-vectors

Definition 4.16 *A subspace* $\mathcal{M} \subset \mathcal{V}$ *is said to be an* ***invariant subspace*** *of an linear operator* X *if* $\forall f \in \mathcal{M} X f \in \mathcal{M}$.

Definition 4.17 *Let* T *be linear operator. If* f *is a non-zero vector satisfying*

$$Tf = \lambda f$$

or some scalar λ *, we say that* f *is an* ***eigen-vector*** *of operator* T *and* λ *is the corresponding* ***eigen-value.***

Note that $f = 0$ will always satisfy the equation $Tf = \lambda f$ for an arbitrary λ. Therefore, null vector is, by definition, excluded from being an eigen-vector.

It is possible that for a given λ there are more than one eigen-vectors satisfying the eigen-value equation $Tf = \lambda$. Therefore, we define

Definition 4.18 *Let* λ *be an eigen-value of an operator* T. *Let* $\nu(\lambda)$ *denote the number of linearly independent eigen-vectors* $Tx = \lambda x$. *If* $\nu(\lambda) = 1$ *we say that the eigen-value* λ *is* ***non-degenerate****. When* $\nu(\lambda) > 1$, *we say that the eigen-value* λ *is* ***degenerate*** *and the* ***degeneracy*** *of the eigen-value* λ *is defined to be equal to the number of linearly independent eigen-vectors with eigen-value* λ.

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[x] I [x] [x] previc [x] conter
Next: Linear Operators - II **Up:** Linear Operators - Basic **Previous:** Inverse of an Operator **Contents**

Ashok K. Kapoor 2004-02-25

In linear algebra, the **eigenvectors** (from the German *eigen* meaning "inherent, characteristic") of a linear operator are non-zero vectors which, when operated on by the operator, result in a scalar multiple of themselves. The scalar is then called the eigenvalue associated with the eigenvector.

In applied mathematics and physics the eigenvectors of a matrix or a differential operator often have important physical significance. In classical mechanics the eigenvectors of the governing equations typically correspond to natural modes of vibration in a body, and the eigenvalues to their frequencies. In quantum mechanics, operators correspond to observable variables, eigenvectors are also called **eigenstates**, and the eigenvalues of an operator represent those values of the corresponding variable that have non-zero probability of occurring.

Examples

Intuitively, for linear transformations of two-dimensional space $\mathbf{R}^2$, eigenvectors are thus:

- rotation: no eigenvectors
- reflection: eigenvectors are perpendicular and parallel to the line of symmetry, the eigenvalues are -1 and 1, respectively
- scaling: all vectors are eigenvectors, and the eigenvalue is the scale factor
- projection onto a line: eigenvectors with eigenvalue 1 are parallel to the line, eigenvectors with eigenvalue 0 are parallel to the direction of projection

Definition

Formally, we define eigenvectors and eigenvalues as follows: If **A** : *V* -> *V* is a linear operator on some vector space *V*, **v** is a non-zero vector in *V* and *c* is a scalar (possibly zero) such that

\\mathbf{A} \\mathbf{v} = c \\mathbf{v},

then we say that **v** is an eigenvector of the operator **A**, and its associated eigenvalue is c. Note that if **v** is an eigenvector with eigenvalue c, then any non-zero multiple of **v** is also an eigenvector with eigenvalue c. In fact, all the eigenvectors with associated eigenvalue c, together with **0**, form a subspace of *V*, the **eigenspace** for the eigenvalue c.

Finding eigenvectors

For example, consider the matrix

A = \\begin{bmatrix} 0 & 1 & -1 \\\1 & 1 & 0 \\\-1 & 0 & 1 \\end{bmatrix}

which represents a linear operator $\mathbf{R}^3$ -> $\mathbf{R}^3$. One can check that

A \\begin{bmatrix}1 \\\\ 1 \\\\ -1 \\end{bmatrix} = 2 \\begin{bmatrix}1 \\\\ 1 \\\\ -1 \\end{bmatrix}

and therefore 2 is an eigenvalue of **A** and we have found a corresponding eigenvector.

The characteristic polynomial

An important tool for describing eigenvalues of square matrices is the characteristic polynomial: saying that *c* is an eigenvalue of **A** is equivalent to stating that the system of linear equations (**A** - *c***I**) **x** = **0** (where **I** is the identity matrix) has a non-zero solution **x** (namely an eigenvector), and so it is equivalent to the determinant det(**A** - *c* **I**) being zero. The function *p*(*c*) = det(**A** - *c***I**) is a polynomial in *c* since determinants are defined as sums of products. This is the *characteristic polynomial* of **A**; its zeros are precisely the eigenvalues of **A**. If **A** is an *n*-by-*n* matrix, then its characteristic polynomial has degree *n* and **A** can therefore have at most *n* eigenvalues.

Returning to the example above, if we wanted to compute all of **A**'s eigenvalues, we could determine the characteristic polynomial first:

p(x) = \\det(A - xI) = \\det

`\\begin{bmatrix}`

-x & 1 & -1 \\\1 & 1-x & 0 \\\-1 & 0 & 1-x \\end{bmatrix}

: = -x^3 + 2x^2 + x - 2\\ and because p(x) = -(x - 2) (x - 1) (x + 1) we see that the eigenvalues of **A** are 2, 1 and -1.

(In practice, eigenvalues of large matrices are not computed using the characteristic polynomial. Faster and more numerically stable methods are available, for instance the QR decomposition.)

Complex eigenvectors

Note that if **A** is a real matrix, the characteristic polynomial will have real coefficients, but not all its roots will necessarily be real. The complex eigenvalues will all be associated to complex eigenvectors.

In general, if $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are eigenvectors to *different* eigenvalues $\lambda_1, \ldots, \lambda_m$, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are necessarily linearly independent.

The spectral theorem for symmetric matrices states that, if **A** is a real symmetric *n*-by-*n* matrix, then all its eigenvalues are real, and there exist *n* linearly independent eigenvectors for **A** which all have length 1 and are mutually orthogonal.

Our example matrix from above is symmetric, and three mutually orthogonal eigenvectors of **A** are

v_1 = \\begin{pmatrix} 1\\\\ 1\\\\ -1\\end{pmatrix}   v_2 = \\begin{pmatrix} 0\\\\ 1\\\\ 1\\end{pmatrix}   v_3 = \\begin{pmatrix} 2\\\\ -1\\\\ 1\\end{pmatrix}

These three vectors form a basis of $\mathbf{R}^3$. With respect to this basis, the linear map represented by **A** takes a particularly simple form: every vector **x** in $\mathbf{R}^3$ can be written uniquely as \\mathbf{x} = x_1 \\mathbf{v}_1 + x_2 \\mathbf{v}_2 + x_3 \\mathbf{v}_3 and then we have \\mathbf{A x} = 2x_1 \\mathbf{v}_1 + x_2 \\mathbf{v}_2 - x_3 \\mathbf{v}_3.

Infinite-dimensional spaces

The concept of eigenvectors can be extended to linear operators acting on infinite-dimensional Hilbert spaces or Banach spaces.

There are operators on Banach spaces which have no eigenvectors at all. For example, take the bilateral shift on the Hilbert space \\ell^2(\\mathbb{Z}); it is easy to see that any potential eigenvector can't be square-summable, so none exist. However, any bounded linear operator on a Banach space *V* does have non-empty **spectrum**. The spectrum \\sigma(T) of the operator $T : V \to V$ is defined as

\\sigma(T)={\\lambda\\in\\mathbb{C}: (\\lambda 1 - T) is not invertible\\}.

Then \\sigma(T) is a compact set of complex numbers, and it is non-empty. When T is a compact operator (and in particular when T is an operator between finite-dimensional spaces as above), the spectrum of T is the same as the set of its eigenvalues.

The spectrum of an operator is an important property in functional analysis.

In mathematics, **Banach spaces**, named after Stefan Banach who studied them, are one of the central objects of study in functional analysis. Banach spaces are typically infinite-dimensional spaces containing functions.

Definition

Banach spaces are defined as complete normed vector spaces. This means that a Banach space is a vector space V over the real or complex numbers with a norm $||.||$ such that every Cauchy sequence (with respect to the metric $d(x, y) = ||x - y||$) in V has a limit in V.

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Linear Operator

An operator $\tilde{L}$ is said to be linear if, for every pair of functions *f* and *g* and scalar *t*,

$$\tilde{L}(f+g) = \tilde{L}f + \tilde{L}g$$

and

$$\tilde{L}(tf) = t\tilde{L}f.$$

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LINEAR OPERATORS ON MATRICES: PRESERVING SPECTRUM AND DISPLACEMENT STRUCTURE

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Abstract. In this paper we characterize those linear operators on general matrices that preserve singular values and displacement rank. We also characterize those linear operators on Hermitian matrices that preserve eigenvalues and displacement inertia.

Key words. linear operator, displacement structure, Toeplitz

AMS(MOS) subject classification. 15A04

1. Introduction. We introduce some notation to facilitate our discussion.

$$\begin{array}{rcl} C^{m\times n} & := & \text{the set of all } m\times n \text{ complex matrices;} \\ Gl(m) & := & \text{the set of all nonsingular } m\times m \text{ matrices;} \\ Herm(m) & := & \text{the set of all } m\times m \text{ Hermitian matrices;} \\ U(m) & := & \text{the set of all } m\times m \text{ unitary matrices.} \end{array}$$

For $1\leq i\leq m, 1\leq j\leq n$, let E^{ij} denote the $m\times n$ matrix with 0 everywhere except 1 at the (i,j) position. Then $\{E^{ij}\}$ is a basis for $C^{m\times n}$. We also adopt the following notation.

$$\begin{array}{rcl} \text{sing}(A) & := & \text{the singular values of a matrix } A \text{ (including multiplicity);} \\ \text{eigen}(A) & := & \text{the eigenvalues of a Hermitian matrix } A \text{ (including multiplicity);} \\ \text{rank}(A) & := & \text{the rank of a matrix } A; \\ \text{inertia}(A) & := & \text{the inertia of a Hermitian matrix } A. \end{array}$$

For $A\in C^{m\times n}, rank(A)=k$ if and only if A has exactly k nonzero singular values. For $A\in Herm(m), inertia(A)=(p,n,z)$ if and only if A has p positive, n negative and z zero eigenvalues, $m=p+n+z$.

We are interested in the spectral properties of matrices that are Toeplitz or nearly Toeplitz. As a consequence, we are interested in linear operators that preserve these properties. We know of only one previous result in this direction. It is the following theorem due to Chu [1992]. Let E_m denote the $m\times m$ *exchange matrix* defined by

$$E_m(i,j) := \delta(i, m+1-j).$$

where δ denotes the Kronecker delta. For example, when $m=3$,

$$E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We write E in place of E_m when m is easily determined from the context.

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THEOREM 1.1. *Let Q be an $m \times m$ orthogonal matrix. Then the following conditions are equivalent:*

(*) If A is an $m \times m$ symmetric Toeplitz matrix then so is QAQ^T.

(**) The matrix Q is one of the following:

$$\pm I, \ \pm E, \ \pm I', \ \pm I'E$$

where I denotes the $m \times m$ identity matrix and $I' := Diag(-1, (-1)^2, \ldots, (-1)^{m-1})$.

His techniques can be used to characterize nonzero linear operators on Hermitian matrices which preservse both eigenvalues and Toeplitz structure. We shall report results along this line in a future paper.

In this report, we study the nonzero linear operators which preserve spectra and displacement structure. We begin by recalling the relevant definitions. Let Z_m denote the $m \times m$ *(lower) shift matrix* defined by

$$Z_m(i,j) := \delta(i, j+1).$$

For example, when $m = 3$,

$$Z_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We write Z in place of Z_m when m is easily determined from the context. Let ∇ be the linear operator defined by

$$\nabla := C^{m \times n} \longrightarrow C^{m \times n} : X \longrightarrow X - Z_m X Z_n^T.$$

For $A \in C^{m \times n}$, the *displacement rank* of A is defined as

$$\text{dis-rank } (A) := \text{rank } (\nabla.A).$$

In the case $m = n$, ∇ preserves Hermitian matrices. For $A \in Herm(m)$, the *displacement inertia* of A is defined as

$$\text{dis-inertia } (A) := \text{inertia } (\nabla.A).$$

Kailath appears to be one of the first to emphasize the importance of the displacement structure of matrices. We recall a few of the major results in this area in order to illustrate the significance of these concepts. Note that Toeplitz matrices usually have displacement rank 2. Hence matrices with low displacement rank are regarded as being "nearly Toeplitz". The following result shows that displacement rank is preserved (loosely speaking) under inversion. It is from Kailath, Kung and Morf [1979].

THEOREM 1.2. *For $A \in Gl(m)$, dis-rank(A^{-1}) = dis-rank(EAE).*

The following inequality, due to Comon [1992], shows that if A has small displacement rank then so does its pseudo-inverse A^+ :

$$\textit{dis-rank}(A^+) \leq 2 \ \textit{dis-rank}(EAE).$$

Note that Hermitian Toeplitz matrices usually have displacement inertia $(1, 1, m-2)$. Hence Hermitian matrices with low displacement inertia are regarded as being "nearly Toeplitz". Similar to displacement rank, displacement inertia is preserved

(loosely speaking) under inversion. We learned about this theorem from Tiberiu Constantinescu (Institute of Mathematics of the Romanian Academy of Sciences).

THEOREM 1.3. *For* $A \in Gl(m) \cap Herm(m)$, $dis\text{-}inertia(A^{-1}) = dis\text{-}inertia(EAE)$. Other versions of displacement structure can be defined and theorems analogous to the last two can often be proved too. See Chun and Kailath [1991], Heinig and Rost [1984].

The rest of this paper is organized as follows. In section 2, we shall characterize those linear operators on general matrices that preserve both rank and displacement rank. As a consequence, we obtain the characterization of those linear operators preserving singular values and displacement rank. The aim of section 3 is to characterize those linear operators on Hermitian matrices that preserve inertia and displacement inertia. We also obtain the characterization of those linear operators preserving eigenvalues and displacement inertia. Then we have some concluding remarks in the final section.

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2. Preserving rank and displacement rank. In this section, we shall characterize those nonzero linear operators on $C^{m\times n}$ that preserve both rank and displacement rank. We shall also characterize those that preserve singular values and displacement rank. Recall that two matrices $A, B \in C^{m\times n}$ are *equivalent* if there exist $M \in Gl(m), N \in Gl(n)$ such that $B = MAN$. Note that A and B are equivalent if and only if $rank(A) = rank(B)$. The following theorem appears in Horn, Li and Tsing [1991]. It characterizes the linear operators preserving equivalence.

THEOREM 2.1. *Let* $T : C^{m\times n} \longrightarrow C^{m\times n}$ *be a nonzero linear operator. Then the following conditions are equivalent:*

(*) T.A is equivalent to T.B whenever A is equivalent to B.

(**) There exist $M \in Gl(m), N \in Gl(n)$ *such that either, for all* $X \in C^{m\times n}, T.X = MXN$ *or m=n and, for all* $X \in C^{m\times n}$, $T.X = MX^TN$.

As consequences of this result, we obtain the characterization of linear operators preserving rank, and those preserving singular values.

THEOREM 2.2. *Let* $T : C^{m\times n} \longrightarrow C^{m\times n}$ *be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in C^{m\times n}, rank(T.X) = rank(X)$.

(**) There exist $M \in Gl(m), N \in Gl(n)$ *such that either, for all* $X \in C^{m\times n}, T.X = MXN$ *or m=n and, for all* $X \in C^{m\times n}, T.X = MX^TN$.

Proof. $(**) \Rightarrow (*)$. Direct verification. $(*) \Rightarrow (**)$. If T preserves rank then it also preserves equivalence. Hence T has the required forms by Theorem 2.1. □

THEOREM 2.3. *Let* $T : C^{m\times n} \longrightarrow C^{m\times n}$ *be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in C^{m\times n}, sing(T.X) = sing(X)$.

(**) There exist $U \in U(m), V \in U(n)$ *such that either, for all* $X \in C^{m\times n}, T.X = UXV$ *or m=n and, for all* $X \in C^{m\times n}, T.X = UX^TV$.

Proof. $(**) \Rightarrow (*)$. Direct verification. $(*) \Rightarrow (**)$. If T preserves singular values then it also preserves rank. By Theorem 2.2, there exist $M \in Gl(m), N \in Gl(n)$

such that either $T.X = MXN$ or $m = n$ and $T.X = MX^TN$. By the singular value decomposition, $M = U_1\Sigma_1 U_2$ and $N = V_1\Sigma_2 V_2$ where $U_i \in U(m), V_i \in U(n)$, $\Sigma_1 = Diag(a_1, \dots, a_m), \Sigma_2 = Diag(b_1, \dots, b_n)$. We consider the case when $T.X = MXN$. If $X = U_2^* E^{ij} V_1^*$ then $T.X = U_1\Sigma_1 E^{ij}\Sigma_2 V_2$. Since $sing(X) = sing(T.X)$, we have $a_i b_j = 1$. Consequently, $a_1 = \cdots = a_m =: a, b_1 = \cdots = b_n =: b$ and $ab = 1$. This implies that $T.X = UXV$ where $U := U_1U_2$ and $V := V_1V_2$. The proof is similar for the other case. □

Now we characterize those linear operators on $C^{m\times n}$ preserving both rank and displacement rank. For $\lambda \in C$, we shall use $D_n(\lambda)$ to denote the $n \times n$ diagonal matrix with diagonal entries $1, \lambda, \dots, \lambda^{n-1}$; in symbols

$$D_n(\lambda) := Diag(1, \lambda, \dots, \lambda^{n-1}).$$

THEOREM 2.4. *Let $T : C^{m\times n} \longrightarrow C^{m\times n}$ be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in C^{m\times n}, rank(T.X) = rank(X)$ and $dis\text{-}rank(T.X) = dis\text{-}rank(X)$.

(**) There exist $\lambda \neq 0$ and lower triangular Toeplitz matrices $M \in Gl(m), N \in Gl(n)$ such that either, for all $X \in C^{m\times n}, T.X = D_m(\lambda)MXN^TD_n(\lambda^{-1})$ or m=n and, for all $X \in C^{m\times n}, T.X = D_m(\lambda)MX^TN^TD_n(\lambda^{-1})$.

Before we prove this theorem, we need some preliminary lemmas. The first one is a characterization of matrices that nearly commute with the shift matrix.

LEMMA 2.5. *Let $B \in C^{n\times n}$ and $\lambda \neq 0$. Then the following conditions are equivalent:*

(*) $BZ_n = \lambda Z_n B$.

(**) There exists a lower triangular Toeplitz matrix L such that $B = D_n(\lambda)L$.

Proof. First we observe that $D_n(\lambda)Z_n = \lambda Z_n D_n(\lambda)$. $(*) \Rightarrow (**)$. Let $L := D_n(\lambda^{-1})B$. Then $LZ_n = Z_nL$. By comparing entries, one deduces that L is a lower triangular Toeplitz matrix. $(**) \Rightarrow (*)$. Since L is a lower triangular Toeplitz matrix, there exists a polynomial $p(x)$ such that $L = p(Z_n)$. Hence $BZ_n = D_n(\lambda)LZ_n = D_n(\lambda)p(Z_n)Z_n = D_n(\lambda)Z_np(Z_n) = \lambda Z_nD_n(\lambda)p(Z_n) = \lambda Z_nD_n(\lambda)L = \lambda Z_nB$. □

Next we collect some basic results about the Kronecker product. For $A \in C^{n\times n}$ and $B \in C^{m\times m}$, recall that $A \otimes B : C^{m\times n} \longrightarrow C^{m\times n}$ is a linear operator which may be defined by

$$(A \otimes B).X := BXA^T.$$

We prefer this "coordinate-free" definition to the usual one. (Compare Horn and Johnson [1991] or Graham [1981] or Lancaster and Tismenetsky [1985]). A fundamental property (which is easy to verify using this definition) is that

$$(A \otimes B) \circ (C \otimes D) = (AC \otimes BD)$$

where $\circ$ denotes the composition of two operators. Moreover it can be proved that $eigen(A \otimes B) = \{\alpha_i\beta_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ where $eigen(A) = \{\alpha_i : 1 \leq i \leq n\}$ and $eigen(B) = \{\beta_j : 1 \leq j \leq m\}$. Hence $tr(A \otimes B) = (trA)(trB)$. The next result, which is taken from Marcus and Moyls [1959], is a form of uniqueness for Kronecker product representations.

LEMMA 2.6. *Let $X_i, W_i \in C^{n\times n}$ and $Y_i, V_i \in C^{m\times m}$. If $\sum_{i=1}^r X_i \otimes Y_i = \sum_{i=1}^s W_i \otimes V_i$. and the X_i are linearly independent then each $Y_i \in Span\{V_1, \dots, V_s\}$.*

Proof. Since the X_j are linearly independent, for each i there exists P_i such that $tr(P_i X_j) = \delta(i,j)$. Then, by "contraction", we have

$$Y_i = \sum_{j=1}^{s} tr(P_i W_j)\, V_j.$$

□

COROLLARY 2.7. *Let $X_i \in C^{n\times n}$ and $Y_j \in C^{m\times m}$. If $\{X_i\}$ and $\{Y_j\}$ are linearly independent sets of matrices then $\{X_i \otimes Y_j\}$ is a linearly independent set.*

Proof. Assume $\sum_{i,j} a_{ij} X_i \otimes Y_j = 0$. We rewrite this equation as

$$\sum_i [X_i \otimes (\sum_j a_{ij} Y_j)] = 0.$$

Use Lemma 2.6 (with all $W_i = 0$) to conclude that, for all i,

$$\sum_j a_{ij} Y_j = 0.$$

Since the Y_j are linearly independent, $a_{ij} = 0$ for all i, j. □

COROLLARY 2.8. *Let $X \in Gl(n), Y \in Gl(m)$. Then $\{X \otimes Y, Z_n X \otimes Y, X \otimes Z_m Y, Z_n X \otimes Z_m Y\}$ is a linearly independent set.*

Proof. Since $X \in Gl(n)$, $\{X, Z_n X\}$ is a linearly independent set. Similarly $\{Y, Z_m Y\}$ is a linearly independent set. Apply Corollary 2.7 to obtain the required result. □

The following result appears in Horn and Johnson [1991] and Graham [1981].

LEMMA 2.9. *Let $trans := C^{n\times n} \longrightarrow C^{n\times n} : X \longrightarrow X^T$ denote the linear operator of taking transpose. Then trans has the following Kronecker product representation:*

$$trans = \sum_{i,j=1}^{n} E^{ij} \otimes E^{ji}$$

where E^{ij} is the $n \times n$ matrix with 0 everywhere except 1 at the (i,j) position.

Proof. For $X = (x_{ij}) \in C^{n\times n}$, note that $E^{ji} X E^{ji} = x_{ij} E^{ji}$. Now we have

$$trans.X = X^T = \sum_{i,j} x_{ij} E^{ji} = \sum_{i,j} E^{ji} X E^{ji} = \sum_{i,j} E^{ij} \otimes E^{ji}.X.$$

□

We adopt the convention that $E^{ij} = 0$ if $i > n, j > n, i < 1$, or $j < 1$. Then it is easy to verify that $ZE^{ij} = E^{(i+1)j}$, and $E^{ij} Z = E^{i(j-1)}$. With this observation, we are ready to prove the next lemma.

LEMMA 2.10. *If $P, Q, R, S \in C^{n\times n}$ are such that*

$$(I_n \otimes I_n - Z_n \otimes Z_n) \circ (Q \otimes P) = (S \otimes R) \circ trans \circ (I_n \otimes I_n - Z_n \otimes Z_n)$$

then at least one of $\{P, R, S\}$ is singular.

Proof. Assume that $P, R, S \in Gl(n)$. By Lemma 2.9, $trans = \sum_{i,j=1}^{n} E^{ij} \otimes E^{ji}$. Hence

$$(I_n \otimes I_n - Z_n \otimes Z_n)(Q \otimes P) = (S \otimes R)(\sum_{i,j=1}^{n} E^{ij} \otimes E^{ji})(I_n \otimes I_n - Z_n \otimes Z_n).$$

Since

$$\begin{aligned}\sum_{i,j=1}^{n}(E^{ij}\otimes E^{ji})(I_n\otimes I_n - Z_n\otimes Z_n) &= \sum_{i,j=1}^{n} E^{ij}\otimes E^{ji} - E^{i(j-1)}\otimes E^{j(i-1)}\\ &= \sum_{i,j=1}^{n} E^{ij}\otimes E^{ji} - E^{ij}\otimes E^{(j+1)(i-1)}\\ &= \sum_{i,j=1}^{n} E^{ij}\otimes (E^{ji} - E^{(j+1)(i-1)})\end{aligned}$$

we have

$$Q\otimes P - Z_nQ\otimes Z_nP = \sum_{i,j=1}^{n} SE^{ij}\otimes R(E^{ji} - E^{(j+1)(i-1)}).$$

Note that $\{SE^{ij}\}$ is a linearly independent set. Use Lemma 2.6 to conclude that, for all i, j, $R(E^{ji} - E^{(j+1)(i-1)}) \in Span\{P, Z_nP\}$. In particular, $RE^{11}, RE^{21}, R(E^{12} - E^{21}) \in Span\{P, Z_nP\}$. Hence

$$3 = dim\ Span\{RE^{11}, RE^{21}, R(E^{12} - E^{21})\} \leq dim\ Span\{P, Z_nP\} = 2.$$

This is a contraction. □

LEMMA 2.11. *If $P, R \in Gl(m)$ and $Q, S \in Gl(n)$ satisfy*

$$(I_n\otimes I_m - Z_n\otimes Z_m)\circ(Q\otimes P) = (S\otimes R)\circ(I_n\otimes I_m - Z_n\otimes Z_m)$$

then there exist $\lambda \neq 0$ and lower triangular Toeplitz matrices $N \in Gl(n), M \in Gl(m)$ such that $Q = D_n(\lambda^{-1})N$ and $P = D_m(\lambda)M$.

Proof. Note that we can rewrite the given equation as follows:

$$(1)\qquad Q\otimes P - Z_nQ\otimes Z_mP = S\otimes R - SZ_n\otimes RZ_m.$$

By Lemma 2.6, $S, SZ_n \in Span\{Q, Z_nQ\}$, i.e. there exist $\alpha, \beta, \gamma, \delta \in C$ such that

$$S = \alpha Q + \beta Z_nQ \quad\text{and}\quad SZ_n = \gamma Q + \delta Z_nQ.$$

Note that $S = (\alpha I + \beta Z_n)Q$ has rank n; hence $\alpha \neq 0$. Also note that $SZ_n = (\gamma I + \delta Z_n)Q$ has rank $n-1$; hence $\gamma = 0$. Furthermore $0 \neq SZ_n = \delta Z_nQ$ and hence $\delta \neq 0$. In summary, we have

$$S = \alpha Q + \beta Z_nQ \quad\text{and}\quad SZ_n = \delta Z_nQ.$$

where $\alpha \neq 0$ and $\delta \neq 0$. Similarly, we get

$$R = aP + bZ_mP \quad\text{and}\quad RZ_m = dZ_mP$$

where $a \neq 0$ and $d \neq 0$. Substituting back into the equation (1), we deduce that, by Corollary 2.8, $\beta = 0$, $b = 0$ and $\alpha a = \delta d = 1$. Thus

$$S = \alpha Q,\ SZ_n = \delta Z_nQ$$

$$R = aP,\ RZ_m = dZ_mP$$

and hence

$$\lambda^{-1} Z_n Q = Q Z_n \quad \text{and} \quad \lambda Z_m P = P Z_m$$

where $\lambda := \frac{\alpha}{\delta} = \frac{d}{a}$. By Lemma 2.5,

$$Q = D_n(\lambda^{-1}) N \quad \text{and} \quad P = D_m(\lambda) M$$

where M, N are lower triangular Toeplitz matrices of dimension m, n respectively. □

We are now ready to prove Theorem 2.4.

Proof. $(**) \Rightarrow (*)$. It is clear that T preserves rank. It remains to show that T preserves displacement rank.

Case 1. $T.X = D_m(\lambda) M X N^T D_n(\lambda^{-1})$

By Lemma 2.5, we have

$$\begin{aligned} T.X - Z_m(T.X)Z_n^T &= D_m(\lambda) M X N^T D_n(\lambda^{-1}) - Z_m D_m(\lambda) M X N^T D(\lambda^{-1}) Z_n^T \\ &= D_m(\lambda) M X N^T D_n(\lambda^{-1}) - \lambda^{-1} D_m(\lambda) M Z_m X \lambda Z_n^T N^T D(\lambda^{-1}) \\ &= D_m(\lambda) M X N^T D_n(\lambda^{-1}) - D_m(\lambda) M Z_m X Z_n^T N^T D(\lambda^{-1}) \\ &= D_m(\lambda) M (X - Z_m X Z_n^T) N^T D(\lambda^{-1}). \end{aligned}$$

Hence

$$\begin{aligned} \textit{dis-rank}(T.X) &= \textit{rank}(T.X - Z_m(T.X)Z_n^T) \\ &= \textit{rank}(X - Z_m X Z_n^T) \\ &= \textit{dis-rank}(X). \end{aligned}$$

Case 2. $T.X = D_m(\lambda) M X^T N^T D_n(\lambda^{-1})$

Using an argument like Case 1, we conclude that T preserves displacement rank.

$(*) \Rightarrow (**)$. We assume that T is a nonzero linear operator that preserves rank and displacement rank. We define $\hat{T} : C^{m\times n} \longrightarrow C^{m\times n}$ by

$$\hat{T} := (I_n \otimes I_m - Z_n \otimes Z_m^T) \circ T \circ (I_n \otimes I_m - Z_n \otimes Z_m^T)^{-1}.$$

Hence

$$(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ T = \hat{T} \circ (I_n \otimes I_m - Z_n \otimes Z_m^T).$$

Since T preserves rank, by Theorem 2.2, there exist $P \in Gl(m), Q \in Gl(n)$ such that either $T = Q \otimes P$ or $m = n$ and $T = (Q \otimes P) \circ \textit{trans}$. On the other hand, since T preserves displacement rank, it follows that $\hat{T}$ preserve rank. Then, by Theorem 2.2, there exist $R \in Gl(m), S \in Gl(n)$ such that either $\hat{T} = S \otimes R$ or $m = n$ and $\hat{T} = (S \otimes R) \circ \textit{trans}$. We have 4 cases to consider.

Case 1. $T = Q \otimes P$ and $\hat{T} = S \otimes R$

Note that $(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ (Q \otimes P) = (S \otimes R) \circ (I_n \otimes I_m - Z_n \otimes Z_m^T)$. Then, by Lemma 2.11, there exist $\lambda \neq 0$ and lower triangular Toeplitz matrices $N \in Gl(n), M \in Gl(m)$ such that $Q = D_n(\lambda^{-1}) N$ and $P = D_m(\lambda) M$. Consequently, $T = D_n(\lambda^{-1}) N \otimes D_m(\lambda) M$.

Case 2. $m = n$, $T = Q \otimes P$ and $\hat{T} = (S \otimes R) \circ \textit{trans}$

Note that $(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ (Q \otimes P) = (S \otimes R) \circ \textit{trans} \circ (I_n \otimes I_m - Z_n \otimes Z_m^T)$. Then, by Lemma 2.10, one of $\{P, R, S\}$ is singular, a contradiction.

Case 3. $m = n$, $T = (Q \otimes P) \circ trans$ and $\hat{T} = S \otimes R$

Note that $(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ (Q \otimes P) \circ trans = (S \otimes R) \circ (I_n \otimes I_m - Z_n \otimes Z_m^T)$. Using the fact that

$$(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ trans = trans \circ (I_n \otimes I_m - Z_n \otimes Z_m^T),$$

we deduce that

$$(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ (Q \otimes P) = (S \otimes R) \circ trans \circ (I_n \otimes I_m - Z_n \otimes Z_m^T).$$

Then, by Lemma 2.10, one of $\{P, R, S\}$ is singular, a contradiction.

Case 4. $m = n$, $T = (Q \otimes P) \circ trans$ and $\hat{T} = (S \otimes R) \circ trans$

Note that $(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ (Q \otimes P) \circ trans = (S \otimes R) \circ trans \circ (I_n \otimes I_m - Z_n \otimes Z_m^T)$. Using the fact that

$$(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ trans = trans \circ (I_n \otimes I_m - Z_n \otimes Z_m^T),$$

we deduce that

$$(I_n \otimes I_m - Z_n \otimes Z_m^T) \circ (Q \otimes P) = (S \otimes R) \circ (I_n \otimes I_m - Z_n \otimes Z_m^T).$$

Then, by Lemma 2.11, there exist $\lambda \neq 0$ and lower triangular Toeplitz matrices $N \in Gl(n), M \in Gl(m)$ such that $Q = D_n(\lambda^{-1})N$ and $P = D_m(\lambda)M$. Consequently, $T = D_n(\lambda^{-1})N \otimes D_m(\lambda)M$. $\square$

$\square$

Next we give the characterization of those linear operators on $C^{n \times n}$ preserving both singular values and displacement rank.

THEOREM 2.12. *Let $T : C^{m \times n} \longrightarrow C^{m \times n}$ be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in C^{m \times n}$, $sing(T.X) = sing(X)$ and dis-rank$(T.X)$ = dis-rank(X).

(**) There exist $|\lambda| = |\mu| = 1$ such that either, for all $X \in C^{m \times n}, T.X = \mu D_m(\lambda) X D_n(\lambda^{-1})$ or m=n and, for all $X \in C^{m \times n}, T.X = \mu D_m(\lambda) X^T D_n(\lambda^{-1})$.

Proof. $(**) \Rightarrow (*)$. Since $|\lambda| = |\mu| = 1, \mu D_m(\lambda)$ and $D_n(\lambda^{-1})$ are unitary. Hence T preserves singular values. By Theorem 2.4, we know T also preserves displacement rank. $(*) \Rightarrow (**)$. Since T preserves singular values, by Theorem 2.3, there exist $U \in U(m), V \in U(n)$ such that either $T.X = UXV$ or $m = n$ and $T.X = UX^TV$. On the other hand, since T preserves both rank and displacement rank, by Theorem 2.4, there exist $\lambda \neq 0$ and lower triangular Toeplitz matrices $M \in Gl(m), N \in Gl(n)$ such that either $T.X = D_m(\lambda)MXN^TD_n(\lambda^{-1})$ or $m = n$ and $T.X = D_m(\lambda)MX^TN^TD_n(\lambda^{-1})$. We consider the following 4 cases.

Case 1. $T.X = D_m(\lambda)MXN^TD_n(\lambda^{-1})$ and $T.X = UXV$.

For all $X \in C^{m \times n}, D_m(\lambda)MXN^TD_n(\lambda^{-1}) = UXV$. Then there exists $\alpha \in C$ such that $\alpha D_m(\lambda)M = U$ and $\frac{1}{\alpha}N^TD_n(\lambda^{-1}) = V$. Therefore both $\alpha D_m(\lambda)M$ and $\frac{1}{\alpha}N^TD_n(\lambda^{-1})$ are diagonal and so $M = uI_m$ and $N = vI_n$ for some $u, v \in C$. Moreover $|\lambda| = |\alpha| = |uv| = 1$. Consequently, $T.X = \mu D_m(\lambda)XD_n(\lambda^{-1})$ where $\mu := uv$.

Case 2. $m = n$, $T.X = D_m(\lambda)MXN^TD_n(\lambda^{-1})$ and $T.X = UX^TV$.

For all $X \in C^{m \times n}, D_m(\lambda)MXN^TD_n(\lambda^{-1}) = UX^TV$. Evaluating at $X = I_n$, we get $D_m(\lambda)MN^TD_n(\lambda^{-1}) = UV$. Let $W := U^*D_m(\lambda)M = VD_n(\lambda)N^{-T}$. Then $WX = X^TW$ for all $X \in C^{m \times n}$. In particular, W commutes with every diagonal matix. Hence W is a scalar, and so $X = X^T$ for all $X \in C^{m \times n}$, a contradiction.

Case 3. $m = n$, $T.X = D_m(\lambda)MX^TN^TD_n(\lambda^{-1})$ and $T.X = UXV$.

Using the same argument as in Case 2, we conclude that case 3 is impossible.

Case 4. $m = n$, $T.X = D_m(\lambda)MX^TN^TD_n(\lambda^{-1})$ and $T.X = UX^TV$.

Using the same argument as in Case 1, we conclude that $T.X = \mu D_m(\lambda)X^TD_n(\lambda^{-1})$. $\square$

3. Preserving inertia and displacement inertia. In this section, we shall characterize those nonzero linear operators on $Herm(n)$ that preserve both inertia and displacement inertia. We shall also characterize those preserving eigenvalues and displacement inertia. Recall that two matrices $A, B \in Herm(n)$ are *$*$-congruent* if there exists $S \in Gl(n)$ such that $B = SAS^*$. Note that A and B are $*$-congruent if and only if $inertia(A) = inertia(B)$. The following theorem appears in Horn, Li and Tsing [1991]. It characterizes the linear operators preserving $*$-congruence. For simplicity, we write I_n as I, Z_n as Z and $D_n(\lambda)$ as $D(\lambda)$ in the following.

THEOREM 3.1. *Let $T : Herm(n) \longrightarrow Herm(n)$ be a nonzero linear operator. Then the following conditions are equivalent:*

(*) $T.A$ is $*$-congruent to $T.B$ whenever A is $*$-congruent to B.

(**) There exists $S \in Gl(n)$ such that either, for all $X \in Herm(n), T.X = \pm SXS^*$ or, for all $X \in Herm(n), T.X = \pm SX^TS^*$.

As consequences of this result, we obtain the characterization of linear operators preserving inertia and those preserving eigenvalues.

THEOREM 3.2. *Let $T : Herm(n) \longrightarrow Herm(n)$ be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in Herm(n), inertia(T.X) = inertia(X)$.

(**) There exists $S \in Gl(n)$ such that either, for all $X \in Herm(n), T.X = SXS^*$ or, for all $X \in Herm(n), T.X = SX^TS^*$.

Proof. $(**) \Rightarrow (*)$. Direct verification. $(*) \Rightarrow (**)$. If T preserves inertia then it also preserves $*$-congruence. Hence, by Theorem 3.1, there exists $S \in Gl(n)$ such that either $T.X = \pm SXS^*$ or $T.X = \pm SX^TS^*$. However the cases with minus signs are ruled out because T preserves inertia. $\square$

THEOREM 3.3. *Let $T : Herm(n) \longrightarrow Herm(n)$ be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in Herm(n), eigen(T.X) = eigen(X)$.

(**) There exists $U \in U(n)$ such that either, for all $X \in Herm(n), T.X = UXU^*$ or, for all $X \in Herm(n), T.X = UX^TU^*$.

Proof. $(**) \Rightarrow (*)$. Direct verification. $(*) \Rightarrow (**)$. If T preserves eigenvalues then it also preserves inertia. Hence, by Theorem 3.2, there exists $S \in Gl(n)$ such that either $T.X = SXS^*$ or $T.X = SX^TS^*$. Since T preserves eigenvalues, $eigen(I) = eigen(T.I) = eigen(SS^*)$. Therefore $SS^* = I$ and so $S \in U(n)$. $\square$

Now we characterize those linear operators on $Herm(n)$ preserving both inertia and displacement inertia.

THEOREM 3.4. *Let $T : Herm(n) \longrightarrow Herm(n)$ be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in Herm(n), inertia(T.X) = inertia(X)$ and $dis\text{-}inertia(T.X) = dis\text{-}inertia(X)$.

(**) There exists $|\lambda| = 1$ and a lower triangular Toeplitz $N \in Gl(n)$ such that either, for all $X \in Herm(n), T.X = D(\lambda)NXN^*D(\lambda)^*$ or, for all $X \in Herm(n), T.X = D(\lambda)NX^TN^*D(\lambda)^*$.

Proof. $(**) \Rightarrow (*)$. It is clear that T preserves inertia. It remains to show that T preserves displacement inertia.

Case 1. $T.X = D(\lambda)NXN^*D(\lambda)^*$

By Lemma 2.5, we have

$$\begin{aligned} T.X - Z(T.X)Z^T &= D(\lambda)NXN^*D(\lambda)^* - ZD(\lambda)NXN^*D(\lambda)^*Z^T \\ &= D(\lambda)NXN^*D(\lambda)^* - \lambda^{-1}D(\lambda)NZX(\lambda^*)^{-1}Z^TN^*D(\lambda)^* \\ &= D(\lambda)NXN^*D(\lambda)^* - D(\lambda)NZXZ^TN^*D(\lambda)^* \\ &= D(\lambda)N(X - ZXZ^T)N^*D(\lambda)^* \end{aligned}$$

Hence

$$\begin{aligned} \textit{dis-inertia}(T.X) &= \textit{inertia}(T.X - Z(T.X)Z^T) \\ &= \textit{inertia}(X - ZXZ^T) \\ &= \textit{dis-inertia}(X). \end{aligned}$$

Case 2. $T.X = D(\lambda)NX^TN^*D(\lambda)^*$

Using an argument like Case 1, we conclude that T preserves displacement inertia.

$(*) \Rightarrow (**)$. We assume that T is a nonzero linear operator that preserves inertia and displacement inertia. We define $\hat{T} : Herm(n) \longrightarrow Herm(n)$ by

$$\hat{T} := (I \otimes I - Z \otimes Z^T) \circ T \circ (I \otimes I - Z \otimes Z^T)^{-1}.$$

Hence

$$(I \otimes I - Z \otimes Z^T) \circ T = \hat{T} \circ (I \otimes I - Z \otimes Z^T).$$

Since T preserves inertia, by Theorem 3.2, there exist $S \in Gl(n)$ such that either $T = \bar{S} \otimes S$ or $T = (\bar{S} \otimes S) \circ \textit{trans}$. On the other hand, since T preserves displacement inertia, it follows that $\hat{T}$ preserve inertia. Then, by Theorem 3.2, there exist $R \in Gl(n)$ such that either $\hat{T} = \bar{R} \otimes R$ or $\hat{T} = (\bar{R} \otimes R) \circ \textit{trans}$. We have 4 cases to consider.

Case 1. $T = \bar{S} \otimes S$ and $\hat{T} = \bar{R} \otimes R$

Note that $(I \otimes I - Z \otimes Z^T) \circ (\bar{S} \otimes S) = (\bar{R} \otimes R) \circ (I \otimes I - Z \otimes Z^T)$. Then, by Lemma 2.11, there exist $\lambda \neq 0$ and lower triangular Toeplitz matrices $N, M \in Gl(n)$ such that $\bar{S} = D(\lambda^{-1})M$ and $S = D(\lambda)N$. Hence $\bar{M} = D(|\lambda|^2)N$, and so $|\lambda| = 1$. Consequently, $T.X = D(\lambda)NX(D(\lambda)N)^*$ for all $X \in Herm(n)$.

Case 2. $T = \bar{S} \otimes S$ and $\hat{T} = (\bar{R} \otimes R) \circ \textit{trans}$

Note that $(I \otimes I - Z \otimes Z^T) \circ (\bar{S} \otimes S) = (\bar{R} \otimes R) \circ \textit{trans} \circ (I \otimes I - Z \otimes Z^T)$. By Lemma 2.10, we get a contradiction.

Case 3. $T = (\bar{S} \otimes S) \circ \textit{trans}$ and $\hat{T} = \bar{R} \otimes R$

Note that $(I \otimes I - Z \otimes Z^T) \circ (\bar{S} \otimes S) \circ \textit{trans} = (\bar{R} \otimes R) \circ (I \otimes I - Z \otimes Z^T)$. Using the fact that

$$(I \otimes I - Z \otimes Z^T) \circ \textit{trans} = \textit{trans} \circ (I \otimes I - Z \otimes Z^T),$$

we deduce that

$$(I \otimes I - Z \otimes Z^T) \circ (\bar{S} \otimes S) = (\bar{R} \otimes R) \circ \textit{trans} \circ (I \otimes I - Z \otimes Z^T),$$

which leads to a contradiction by Lemma 2.10.

Case 4. $T = (\bar{S} \otimes S) \circ trans$ and $\hat{T} = (\bar{R} \otimes R) \circ trans$

Note that $(I \otimes I - Z \otimes Z^T) \circ (\bar{S} \otimes S) \circ trans = (\bar{R} \otimes R) \circ trans \circ (I \otimes I - Z \otimes Z^T)$. Using the fact that

$$(I \otimes I - Z \otimes Z^T) \circ trans = trans \circ (I \otimes I - Z \otimes Z^T),$$

we deduce that

$$(I \otimes I - Z \otimes Z^T) \circ (\bar{S} \otimes S) = (\bar{R} \otimes R) \circ (I \otimes I - Z \otimes Z^T).$$

Then we obtain the required result as in Case 1.

□

Next we give the characterization of those linear operators on $Herm(n)$ preserving both eigenvalues and displacement inertia.

THEOREM 3.5. *Let $T : Herm(n) \longrightarrow Herm(n)$ be a nonzero linear operator. Then the following conditions are equivalent:*

(*) For all $X \in Herm(n), eigen(T.X) = eigen(X)$ and dis-inertia$(T.X)$ = dis-inertia(X).

(**) There exists $|\lambda| = 1$ such that either, for all $X \in Herm(n), T.X = D(\lambda) X D(\lambda)^*$ or, for all $X \in Herm(n), T.X = D(\lambda) X^T D(\lambda)^*$.

Proof. $(**) \Rightarrow (*)$. Since $|\lambda| = 1$, $D(\lambda)$ is unitary and hence T preserves eigenvalues. From Theorem 3.4, it is clear that T also preserves displacement inertia. $(*) \Rightarrow (**)$. Since T preserves eigenvalues, from Theorem 3.3, there exists $U \in U(n)$ such that either $T.X = UXU^*$ or $T.X = UX^TU^*$. On the other hand, since T preserves both inertia and displacement inertia, from Theorem 3.4, there exists $|\lambda| = 1$ and a lower triangular Toeplitz $N \in Gl(n)$ such that either $T.X = D(\lambda)NXN^*D(\lambda)^*$ or $T.X = D(\lambda)NX^TN^*D(\lambda)^*$. We consider the following 4 cases.

Case 1. $T.X = D(\lambda)NXN^*D(\lambda)^*$ and $T.X = UXU^*$.

Evaluating at $X = I$, we get $D(\lambda)NN^*D(\lambda)^* = UU^* = I$, i.e. $D(\lambda)N$ is unitary. Since $D(\lambda)N$ is lower triangular, it must be diagonal and so is N. This implies that $N = \mu I$ because N is Toeplitz. Moreover $|\mu| = 1$. Finally $T.X = D(\lambda)XD(\lambda)^*$.

Case 2. $T.X = D(\lambda)NXN^*D(\lambda)^*$ and $T.X = UX^TU^*$.

For all $X \in Herm(n)$, $D(\lambda)NXN^*D(\lambda)^* = UX^TU^*$. Evaluating at $X = I$, we find that $D(\lambda)N$ is unitary. Let $V := U^*D(\lambda)N$. Then $VX = X^TV$ for all $X \in Herm(n)$. This implies that V must be a scalar, and so $X = X^T$ for all $X \in Herm(n)$, a contradiction.

Case 3. $T.X = D(\lambda)NX^TN^*D(\lambda)^*$ and $T.X = UXU^*$.

Using the same argument as in Case 2, we conclude that Case 3 is impossible.

Case 4. $T.X = D(\lambda)NX^TN^*D(\lambda)^*$ and $T.X = UX^TU^*$.

Using the same argument as in Case 1, we conclude that $T.X = D(\lambda)X^TD(\lambda)^*$.

□

4. Concluding Remarks. There are many papers on rank preserving linear operators and inertia preserving linear operators: Adams [1962], Adams, Lax and Phillips [1965], Atkinson [1983], Atkinson and Lloyd [1980,1981], Baruch and Loewy [1991], Beasley [1970,1981,1983,1988], Botta [1987], Eisenbud and Harris [1988], Flanders [1962], Helton and Rodman [1985], Johnson and Pierce [1985,1986], Laffey and Loewy [1990], Lim [1979], Loewy [1989,1990,1991], Loewy and Pierce [1991], Marcus and Moyls [1959], Meshulam [1985,1989], Moore [1966], Pierce and Rodman [1988], Schneider [1965], Sylvester [1986], Westwick [1967,1987]. (We received this list from

R. Loewy. We have not looked at all these papers.) Some of them characterize linear preservers of one particular rank class or one particular inertia class (rather than characterizing preservers of all rank or inertia classes as was done in Theorem 2.2 and 3.2). These results probably make it possible to characterize linear preservers of one particular rank-and-displacement-rank class and linear preservers of one particular inertia-and displacement-inertia class. Many of the results in these references treat rank preservers or inertia preservers over the field of real numbers (rather than the field of complex numbers that we used in this report). Some of the references even deal with more general fields of numbers. These preserver results over other fields probably make it possible to extend the results of this report to other fields of numbers.

We mentioned earlier that there are definitions of displacement structure that are different than the ones we use in this report. (See Chu and Kailath [1991] and Heinig and Rost [1984].) There are linear preserver questions analogous to the ones we studied here for the other definitions. We expect that the techniques that we have used here can be used to easily settle such analogous questions.

In the introduction of this report we noted that we are interested in the spectral properties of matrices that are Toeplitz or nearly Toeplitz. In particular, we are interested in sets having the following forms

$$eigen^{-1}(\lambda) \cap Toep(m)$$

or

$$eigen^{-1}(\lambda) \cap dis\text{-}inertia^{-1}(p,n,z)$$

where

$$\begin{aligned} \lambda &:= (\lambda_1, \ldots, \lambda_m) \in R^m, \\ eigen^{-1}(\lambda) &:= \{A \in Herm(m) : eigen(A) = \lambda\}, \\ Toep(m) &:= \{A \in C^{m\times m} : A \text{ is Toeplitz}\}, \\ dis\text{-}inertia^{-1}(p,n,z) &:= \{A \in Herm(m) : dis\text{-}inertia(A) = (p,n,z)\}. \end{aligned}$$

Now, by the spectral theorem, we have that

$$eigen^{-1}(\lambda) = \{QDiag(\lambda)Q^* : Q \in U(m)\}.$$

¿From this we see that linear spectra preserving operators

$$Herm(m) \longrightarrow Herm(m) : X \longrightarrow QXQ^*$$

for $Q \in U(m)$ can be used to move around this isospectral surface $eigen^{-1}(\lambda)$. In more technical language, we see that $eigen^{-1}(\lambda)$ is the orbit of $Diag(\lambda)$ under the group action defined by

$$U(m) \times Herm(m) \longrightarrow Herm(m) : (Q,X) \longrightarrow QXQ^*.$$

We originally hoped that we could move around somewhat freely on the sets of the form $eigen^{-1}(\lambda) \cap dis\text{-}inertia^{-1}(p,n,z)$ by means of the linear preservers of such sets. This hope motivated our study of linear preservers. Unfortunately, our hope was too optimistic. Our results show that there are not enough such linear preservers.

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09/ 676,801

CROSS-REFERENCE TO RELATED APPLICATION

This application is a non-provisional application of provisional application Serial No. 60/158,156 filed October 8, 1999.

What is claimed is:

1. A method for determining configuration parameters describing a physical system, the method **comprising** the steps of

measuring an output signal from the system in response to an input signal,

the output signal being related to the configuration parameters

by a linear operator,

and directly reconstructing each of the configuration parameters

by applying a prescribed mathematical algorithm

to the output signal.

2. The method as recited in claim 1 wherein

step of directly reconstructing includes the step of

computing a configuration parameter function.

3. The method as recited in claim 1 wherein

step of directly reconstructing includes the step of

computing a configuration kernel.

4. The method as recited in claim 1 wherein

step of directly reconstructing includes the step of

computing a configuration parameter response function for

each of the configuration parameters.

5. A method for estimating a loop composition in terms of loop parameters representative of the loop composition **comprising** the steps of

energizing the loop from a measurement end with an energy source,

measuring a response signal from the loop at the measurement end,

wherein each of the loop parameters is related to response signal by a linear operator, and

directly reconstructing each of the loop parameters

by

executing a prescribed mathematical algorithm,

determined with reference to the linear operator,

on the response signal.

6. The method as recited in claim 5 wherein

step of directly reconstructing includes

the step of computing a loop parameter function.

7. The method as recited in claim 5 wherein

step of directly reconstructing includes

the step of computing a loop kernel.

8. The method as recited in claim 5 wherein

step of directly reconstructing includes

the step of computing a parameter response function for each of the loop parameters.

9. A method for

estimating a loop composition of a subscriber loop in terms of loop parameters $X_1, X_2, \ldots, X_i, \ldots, X_N$, the loop having a frequency-domain response $H(\omega, X_1, X_2, \ldots, X_i, , X_N)$ for the loop parameters, the method **comprising** the steps of

(a) determining a range for each loop parameter X_i,

(b) for each loop parameter X_i ,

generating a frequency-domain loop parameter function $Fx_i(\omega)$

wherein $Fx_i(\omega) = \int_{x_1} \int_{x_2} \dots \int_{x_i} \dots \int_{x_N} X_i\, H(\omega, X_1, X2, \dots, Xi, \dots, X_N)\, dX_1 dX_2 \dots d X_i \dots dX_N$,

(c) generating a loop kernel $k(\omega, \beta)$ for all loop parameters

wherein $k(\omega, \beta) = \int_{x_1} \int_{x_2} \dots \int_{x_N} H(\omega, X_1, X_2, \ , X_N)\, H(\beta, X_1, X_2, \ , X_N)\, dX_1 dX_2\ dX_N, X_1\ X_2\ X_N$

(d) generating a parameter response function $gi(\beta)$

for each loop parameter from the integral relation $Fx_i(\omega) = \int k(\omega, \beta)\, g_i(\beta) d\beta$,

(e) energizing the loop from a measurement end with an energy source,

(f) measuring a response signal $H_R(\omega) = H(\omega, X_1, X_2, \ . . , X_i, \ , X_N)$

for the loop at the measurement end, and

(g) directly determining each loop parameter X_i

from the integral relation $X_i = \int H_R(\beta)\, g_i(\beta) d\beta$.

10. The method as recited in claim 9 wherein step (e) includes

the step of computing the inverse of $k(\omega, \beta)$.

11. The method as recited in claim 9 wherein step (e) includes

the step of computing the inverse of $k(\omega, \beta)$ using singular value decomposition.

12. The method as recited in claim 11 wherein step (f) includes

the step of filtering noise from the response signal.

13. A system for generating the loop composition in terms of loop parameters representative of the loop composition comprising

a source of waves for energizing the loop

from

a measurement end,

a detector for detecting a response signal

from the loop at the measurement 2 end, wherein each of the loop parameters is related to response signal by an integral operator, and

a reconstructor for directly reconstructing each of the loop parameters by executing a prescribed mathematical algorithm, determined with reference to the integral operator, on the response signal.

14. The system as recited in claim 13 wherein reconstructor includes a processor for computing a loop parameter function.

15. The system as recited in claim 13 wherein reconstructor includes a processor for computing a loop kernel.

16. The system as recited in claim 13 wherein reconstructor includes a processor for computing a parameter response function for each of the loop parameters.